

BROWN REPRESENTABILITY DUAL FOR HOMOTOPY CATEGORIES OF COMPLEXES

GEORGE CIPRIAN MODOI

ABSTRACT. We say that a fixed object is a product generator for an additive category, provided that every other object is a direct factor of a product of copies of it. For an additive category with products and images, e.g. a module category, we prove that Brown representability theorem is valid for the dual of homotopy category of complexes over this category if and only if it has a product generator.

INTRODUCTION

Brown representability is a key tool in the theory of triangulated categories. Recall that if \mathcal{K} is a triangulated category with products then \mathcal{K}^o is said to *satisfy Brown representability* if every homological product preserving functor $F : \mathcal{K} \rightarrow \mathcal{A}b$ is representable. Dually \mathcal{K} satisfies Brown representability if every cohomological (contravariant) functor which sends coproducts into products $F : \mathcal{K} \rightarrow \mathcal{A}b$ is representable. Sometime we call these properties as Brown representability for covariant, respectively contravariant functors.

The notion of well-generated triangulated category was introduced by Neeman in his influential book [14], where it is also shown that Brown representability holds for triangulated categories of this type. Prototypes for (algebraic) well-generated triangulated categories are derived categories and their localizations (see [17]). But until recently, just a few was known about Brown representability for homotopy categories of complexes. The articles [18] and [13] are the first which gave some information in this sense. The present work comes to complete the picture started in [13]. To be precise, let R be a ring. We denote by $\mathbf{K}(\text{Mod}(R))$ the homotopy category of complexes of R -modules. In [13] it is shown that $\mathbf{K}(\text{Mod}(R))$ satisfies Brown representability if and only if R is pure-semisimple. But for the dual $\mathbf{K}(\text{Mod}(R))^o$ only one direction was shown: If $\mathbf{K}(\text{Mod}(R))^o$ satisfies Brown representability then $\text{Mod}(R)$ must have a product generator. Note that a product generator of an additive category \mathcal{A} is defined to be an object G with the property that every object of \mathcal{A} is a direct factor of a product of copies of G . The module category over a pure semi-simple ring R satisfies a dual property, namely $\text{Mod}(R) = \text{Add}(G)$ for some $G \in \text{Mod}(R)$, where through $\text{Add}(G)$ we understand the class of all direct summands of direct

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sums of copies of G . The main result in this paper proves the equivalence between the conditions $\mathbf{K}(\text{Mod}(R))^o$ satisfies Brown representability and $\text{Mod}(R)$ has a product generator. Moreover our approach may be easily dualized in order to give (a generalization of) results in [13] about Brown representability for contravariant functors defined on homotopy category of complexes.

The problem of Brown representability for covariant functors is difficult and not completely solved even in the case of well-generated categories. This is the reason for which the method used to prove this kind of result perhaps deserves a few words. In [12] we proved a generalization of Neeman's variant of Brown representability for contravariant functors defined on well-generated triangulated categories. With this aim, we developed a technique, based on the fact that every object of a well-generated category is the homotopy colimit of a tower of objects which is constructed iteratively starting with a set. The whole construction is analogous to the case of an object of an abelian category which is filtered by a set (see [3, Definition 3.1.1]), but as usually, short exact sequences are replaced by triangles. Naturally appeared the question if the construction may be dualized in order to give some information about Brown representability for covariant functors. Strictly in the settings from [12] the answer is probably not, but we adapted here this method and we observed that if $\text{Mod}(R)$ has a product generator, then every complex in $\mathbf{K}(\text{Mod}(R))$ is cofiltered by a set, what roughly means it is isomorphic to the homotopy limit of an inverse tower constructed iteratively starting with a set.

The paper is organized as follows: Section 1 contains a new proof of an old (but not largely known) representability theorem due to Heller, for functors $F : \mathcal{K} \rightarrow \mathcal{A}b$, where \mathcal{K} is a triangulated category with products. Some applications to much recent results are also indicated. Using this, we prove in Section 2 a new representability theorem, supposing in addition that every object of \mathcal{K} is cofiltered by a set. Next Section 3 contains the main result of this paper: We characterize those additive categories \mathcal{A} closed under idempotents and products and possessing images or kernels for which $\mathbf{K}(\mathcal{A})^o$ satisfies Brown representability as being exactly the categories with a product generator. In particular this is true for \mathcal{A} being the module category over a ring R .

1. A NEW PROOF FOR HELLER'S REPRESENTABILITY THEOREM

Consider a preadditive category \mathcal{K} . We write $\mathcal{K}(K, K')$ for the abelian group of morphisms between K and K' in \mathcal{K} . By a *(right) \mathcal{K} -module* we understand a functor $X : \mathcal{K}^o \rightarrow \mathcal{A}b$. In this paper modules will always be at right, so for dealing with a *left \mathcal{K} -module* we have to consider a right \mathcal{K}^o -module, that is a functor $X : \mathcal{K} \rightarrow \mathcal{A}b$. A \mathcal{K} -module is called *finitely presentable* if there is an exact sequence of functors

$$\mathcal{K}(-, K_1) \rightarrow \mathcal{K}(-, K_0) \rightarrow X \rightarrow 0$$

for some $K_0, K_1 \in \mathcal{K}$. We denote $\text{Hom}_{\mathcal{K}}(X, Y)$ the class of all natural transformations between two \mathcal{K} -modules. Generally there is no reason for this class to be a set. However, using Yoneda lemma, we know that the

$\mathrm{Hom}_{\mathcal{K}}(X, Y)$ is actually a set, provided that X is finitely presentable. We consider the category $\mathrm{mod}(\mathcal{K})$ of all finitely presentable \mathcal{K} -modules, having $\mathrm{Hom}_{\mathcal{K}}(X, Y)$ as morphisms spaces, that is $\mathrm{mod}(\mathcal{K})(X, Y) = \mathrm{Hom}_{\mathcal{K}}(X, Y)$ for all $X, Y \in \mathrm{mod}(\mathcal{K})$.

The Yoneda functor

$$H = H_{\mathcal{K}} : \mathcal{K} \rightarrow \mathrm{mod}(\mathcal{K}^o)^o \text{ given by } H_{\mathcal{K}}(K) = \mathcal{K}(K, -)$$

is an embedding of \mathcal{K} into $\mathrm{mod}(\mathcal{K}^o)^o$, according to Yoneda lemma. If, in addition, \mathcal{K} has products then $\mathrm{mod}(\mathcal{K}^o)^o$ is complete and the Yoneda embedding preserves products. It is also well-known (and easy to prove) that, if $F : \mathcal{K} \rightarrow \mathcal{A}$ is a functor into an additive category with kernels, then there is a unique, up to a natural isomorphism, left exact functor $F^* : \mathrm{mod}(\mathcal{K}^o)^o \rightarrow \mathcal{A}$, such that $F = F^* H_{\mathcal{K}}$ (see [9, Lemma A.1]). Moreover, F preserves products if and only if F^* preserves limits.

Let $F : \mathcal{K} \rightarrow \mathcal{A}b$ be a functor. The *category of elements of F* , denoted by \mathcal{K}/F , has as objects pairs of the form (X, x) where $X \in \mathcal{K}$ and $x \in F(X)$, and a map between (X, x) and (Y, y) in \mathcal{K}/F is a map $f : X \rightarrow Y$ in \mathcal{K} such that $F(f)(x) = y$. Recall that the solution set condition for functors with values in the category of abelian groups $F : \mathcal{K} \rightarrow \mathcal{A}b$ may be stated as follows: there is a set \mathcal{S} of objects in \mathcal{K} , and such that for any $K \in \mathcal{K}$ and any $y \in F(K)$ there are $S \in \mathcal{S}$, $x \in F(S)$ and $f : S \rightarrow K$ such that $F(f)(x) = y$ (see [10, Chapter V, §6, Theorem 3]). We may reformulate this by saying that the category

$$\mathcal{S}/F = \{(S, x) \mid S \in \mathcal{S}, x \in F(S)\}$$

is *weakly initial* in \mathcal{K}/F , that is for every $(K, y) \in \mathcal{K}/F$ there exists a map $(S, x) \rightarrow (K, y)$ for some $(S, x) \in \mathcal{S}/F$. Via Yoneda lemma, every object $(S, x) \in \mathcal{S}/F$ corresponds to a natural transformation $\mathcal{K}(S, -) \rightarrow F$. In these terms, the existence of a solution set is further equivalent to the fact that there are objects $S_i \in \mathcal{K}$ indexed over a set I and a functorial epimorphism

$$\bigoplus_{i \in I} \mathcal{K}(S_i, -) \rightarrow F \rightarrow 0.$$

We say that F has a solution object provided that there is an object $S \in \mathcal{K}$ and a functorial epimorphism

$$\mathcal{K}(S, -) \rightarrow F \rightarrow 0,$$

or equivalently, the category \mathcal{K}/F has a weakly initial object. Note that if there are arbitrary products in \mathcal{K} , and the functor F preserves them, then the existence of a solution set is clearly equivalent to those of a solution object. Obviously if $F \cong \mathcal{K}(S, -)$ is representable, then F has a solution object.

In this section the category \mathcal{K} will be triangulated with splitting idempotents. For definition and basic properties of triangulated categories the standard reference is [14]. Note that \mathcal{K} has splitting idempotents, provided that \mathcal{K} has countable coproducts or products, according to [14, Proposition 1.6.8] or its dual. Recall that \mathcal{K} is supposed to be additive. A functor $\mathcal{K} \rightarrow \mathcal{A}$ into an abelian category \mathcal{A} is called *homological* if it sends triangles into exact sequences. A contravariant functor $\mathcal{K} \rightarrow \mathcal{A}$ which is homological

regarded as a functor $\mathcal{K}^o \rightarrow \mathcal{A}$ is called *cohomological* (see [14, Definition 1.1.7 and Remark 1.1.9]). An example of a homological functor is the Yoneda embedding $H_{\mathcal{K}} : \mathcal{K} \rightarrow \text{mod}(\mathcal{K}^o)^o$. We know that in this case $\text{mod}(\mathcal{K}^o)^o$ is equivalent to $\text{mod}(\mathcal{K})$ (see [14, Remark 5.1.19 and what follows]). Moreover it is an abelian category, and for every functor $F : \mathcal{K} \rightarrow \mathcal{A}$ into an abelian category, the unique left exact functor $F^* : \text{mod}(\mathcal{K}^o)^o \rightarrow \mathcal{A}$ extending F is exact if and only if F is homological, by the dual of [8, Lemma 2.1]. Note that this is the reason for which $\text{mod}(\mathcal{K}^o)^o$ (or often the equivalent category $\text{mod}(\mathcal{K})$) is called the *abelianization* of the triangulated category \mathcal{K} . By [14, Corollary 5.1.23], $\text{mod}(\mathcal{K}^o)^o$ is a Frobenius abelian category, with enough injectives and enough projectives, which are, up to isomorphism, exactly objects of the form $\mathcal{K}(K, -)$ for some $K \in \mathcal{K}$.

Observe that in the particular case when the codomain of the homological functor F is the category $\mathcal{A}b$ of all abelian groups, then it may be easily seen that $F^*(X) \cong \text{Hom}_{\mathcal{K}^o}(X, F)$, naturally for all $X \in \text{mod}(\mathcal{K}^o)^o$. Thus we obtain:

Lemma 1.1. *If \mathcal{K} is a triangulated category with splitting idempotents, then a homological functor $F : \mathcal{K} \rightarrow \mathcal{A}b$ is representable if and only if its extension $F^* : \text{mod}(\mathcal{K}^o)^o \rightarrow \mathcal{A}b$ is representable.*

Proof. If F is representable, then $F \in \text{mod}(\mathcal{K}^o)^o$, so $F^*(X) \cong \text{Hom}_{\mathcal{K}^o}(X, F)$, for all $X \in \text{mod}(\mathcal{K}^o)^o$ and F^* is represented by F . Conversely if F^* is representable by an object in $\text{mod}(\mathcal{K}^o)^o$ then this object must be isomorphic to F , therefore $F \in \text{mod}(\mathcal{K}^o)^o$. Because F^* is exact, F must be projective, hence representable. \square

Lemma 1.2. *If \mathcal{K} is a triangulated category with splitting idempotents, then a cohomological functor $F : \mathcal{K} \rightarrow \mathcal{A}b$ has a solution object if and only if $F^* : \text{mod}(\mathcal{K}^o)^o \rightarrow \mathcal{A}b$ has a solution object.*

Proof. Suppose F has a solution object, i.e. there is a functorial epimorphism $H(K) = \mathcal{K}(K, -) \rightarrow F \rightarrow 0$, with $K \in \mathcal{K}$. In order to show that F^* has a solution object, we want to show that the induced natural transformation

$$\text{Hom}_{\mathcal{K}^o}(-, H(K)) \rightarrow \text{Hom}_{\mathcal{K}^o}(-, F) \cong F^*$$

is an epimorphism. That is, we want to show that the map

$$\text{Hom}_{\mathcal{K}^o}(X, H(K)) \rightarrow \text{Hom}_{\mathcal{K}^o}(X, F)$$

is surjective, for all $X \in \text{mod}(\mathcal{K}^o)^o$. But every $X \in \text{mod}(\mathcal{K}^o)^o$ admits an embedding $0 \rightarrow X \rightarrow H(U)$, that is an epimorphism from the projective object $H(U)$ to X in the opposite category. Since $H(K) \in \text{mod}(\mathcal{K}^o)^o$ is projective and F^* is exact, we obtain a diagram with exact rows:

$$\begin{array}{ccccccc} \text{Hom}_{\mathcal{K}^o}(H(U), H(K)) & \longrightarrow & \text{Hom}_{\mathcal{K}^o}(X, H(K)) & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & & & \\ \text{Hom}_{\mathcal{K}^o}(H(U), F) & \longrightarrow & \text{Hom}_{\mathcal{K}^o}(X, F) & \longrightarrow & 0 & & \end{array}.$$

By Yoneda lemma we know that the first vertical map is isomorphic to $\mathcal{K}(K, U) \rightarrow F(U)$, hence it is surjective, thus the diagram above proves the direct implication.

Conversely if there is $X \in \text{mod}(\mathcal{K}^o)^o$ and a natural epimorphism

$$\text{Hom}_{\mathcal{K}^o}(-, X) \rightarrow \text{Hom}_{\mathcal{K}^o}(-, F) \rightarrow 0,$$

then let $H(K) \rightarrow X \rightarrow 0$ be an epimorphism in $\text{mod}(\mathcal{K}^o)$ (that is a monomorphism in the opposite direction in $\text{mod}(\mathcal{K}^o)^o$, with $K \in \mathcal{K}$. Consider the composed map

$$\text{Hom}_{\mathcal{K}^o}(-, H(K)) \rightarrow \text{Hom}_{\mathcal{K}^o}(-, X) \rightarrow \text{Hom}_{\mathcal{K}^o}(-, F).$$

Evaluating it at $H(U)$ for an arbitrary $U \in \mathcal{K}$, we obtain a surjective natural map $\mathcal{K}(K, U) \rightarrow F(U)$, hence F has a solution object. \square

Theorem 1.3. [4, Theorem 1.4] *If \mathcal{K} is a triangulated category with products, then a homological products preserving functor $F : \mathcal{K} \rightarrow \mathcal{A}b$ is representable if and only if it has a solution object.*

Proof. Under the hypotheses imposed on \mathcal{K} and F , the abelian category $\text{mod}(\mathcal{K}^o)^o$ is complete and the induced functor $F^* : \text{mod}(\mathcal{K}^o)^o \rightarrow \mathcal{A}b$ preserves limits. Therefore it is representable if and only if it has a solution object, by Freyd's Adjoint Functor Theorem. Thus the conclusion follows by combining Lemmas 1.1 and 1.2. \square

Remark 1.4. Theorem 1.3 says more than the Neeman's Freyd style representability theorem [15, Theorem 1.3]. Indeed the cited result states that if every cohomological functor which sends coproducts into products has a solution objects, then every such a functor is representable, whereas our result involves a fixed functor. However the result is known: It already appeared in Heller's paper [4]. We have just proved the dual version because our argument is different from Heller's one, and it shows us explicitly how the result follows from the celebrated Freyd's Adjoint Functor Theorem.

Using the Theorem above we may improve [12, Theorem 3.7], which is the main result there (for the unexplained terms see [12]):

Corollary 1.5. *Let \mathcal{K} be a triangulated category with coproducts which is \aleph_1 -perfectly generated by a projective class \mathcal{P} . If $F : \mathcal{K} \rightarrow \mathcal{A}b$ is a cohomological functor which sends coproducts in products, such that \mathcal{P}^{*n}/F has a weak terminal object for all $n \in \mathbb{N}$, then F is representable.*

In a particular case, namely in the presence of products, we may derive from the above results the dual of [16, Proposition 1.4]. In order to state this, recall that if \mathcal{K} is a full subcategory of \mathcal{T} then a \mathcal{K} -preenvelope of $T \in \mathcal{T}$ is a morphism $T \rightarrow X_T$ with $X_T \in \mathcal{K}$ such that the induced map $\mathcal{T}(X_T, X) \rightarrow \mathcal{T}(T, X)$ is surjective for all $X \in \mathcal{K}$. Dually we define the concept of *precover*. The subcategory \mathcal{K} is called preenveloping is every object in \mathcal{T} admits a \mathcal{K} -preenvelope.

Corollary 1.6. *Let \mathcal{T} be a triangulated category with products, and let \mathcal{K} be a colocalizing subcategory. The following are equivalent:*

- (i) *The inclusion $\mathcal{K} \rightarrow \mathcal{T}$ has a left adjoint.*
- (ii) *Every object in \mathcal{T} admits a \mathcal{K} -preenvelope.*

Proof. Since the implication (i) \Rightarrow (ii) follows from the general theory of adjoint functors, we need to show only the converse. But this follows immediately from Theorem 1.3 since, if $I : \mathcal{K} \rightarrow \mathcal{T}$ is the inclusion functor, then

for every $T \in \mathcal{T}$ the functor $\mathcal{T}(T, I(-)) : \mathcal{K} \rightarrow \mathcal{A}b$ is homological, preserves products and has a solution object, given by the functorial epimorphism $\mathcal{K}(X_T, -) \rightarrow \mathcal{T}(T, I(-))$, where $T \rightarrow X_T$ is a \mathcal{K} -preenvelope of X . \square

2. COFILTERED OBJECTS IN TRIANGULATED CATEGORIES

As before we denote by \mathcal{K} a triangulated category with products. Let $\mathcal{S} \subseteq \mathcal{K}$ be a set of objects. We denote $\text{Prod}_0(\mathcal{S}) = \text{Prod}(\mathcal{S})$ and we define inductively $\text{Prod}_n(\mathcal{S})$ to be the full subcategory of \mathcal{K} which consists of all objects Y lying in a triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ with $X \in \text{Prod}_0(\mathcal{S})$ and $Y \in \text{Prod}_n(\mathcal{S})$. We suppose that \mathcal{S} is closed under suspensions and desuspensions, so the same is true for $\text{Prod}_n(\mathcal{S})$, by [15, Remark 07]. Moreover the same [15, Remark 07] tells us that if $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is a triangle with $X \in \text{Prod}_n(\mathcal{S})$ and $\text{Prod}_m(\mathcal{S})$ then $Z \in \text{Prod}_{n+m}(\mathcal{S})$. We say that an object $X \in \mathcal{K}$ is \mathcal{S} -cofiltered if it may be written as a homotopy limit $X = \varprojlim X_n$ of an inverse tower, with $X_0 \in \text{Prod}_0(\mathcal{S})$, and X_{n+1} lying in a triangle $P_n \rightarrow X_{n+1} \rightarrow X_n \rightarrow P_n[1]$, for some $P_n \in \text{Prod}_0(\mathcal{S})$. Inductively we have $X_n \in \text{Prod}_n(\mathcal{S})$, for all $n \in \mathbb{N}$.

Lemma 2.1. *Let \mathcal{K} be a triangulated category and let $\mathcal{S} \subseteq \mathcal{K}$ be a set closed under suspensions and desuspensions. Suppose that every $X \in \mathcal{K}$ is \mathcal{S} -cofiltered. Then every homological product preserving functor $F : \mathcal{K} \rightarrow \mathcal{A}b$ has a solution object.*

Proof. By [15, Lemma 2.3], we know that the restricted functor $\text{Prod}_n(\mathcal{S}) \rightarrow \mathcal{A}b$, $X \mapsto F(X)$ has a solution object for all $n \in \mathbb{N}$. Now we have to apply the dual of the argument used in the proof of [12, Proposition 3.6]. Since the hypotheses are slightly modified, we sketch here this argument (in the dual form appropriate to the present approach).

For any $n \in \mathbb{N}$, the category $\text{Prod}_n(\mathcal{S})/F$ has a weakly initial object denoted (T_n, t_n) , (which corresponds to a solution object). Let I be the non-empty set of all inverse towers of the form

$$T_0 \xleftarrow{w_0} T_1 \xleftarrow{w_1} T_2 \longleftarrow \dots$$

with $F(w_n)(t_{n+1}) = t_n$, for all $n \in \mathbb{N}$, and denote by $T(i)$ the homotopy limit of the tower $i \in I$. By [1, Lemma 5.8(2)], there is an exact sequence

$$0 \rightarrow \varprojlim^{(1)} F(T_n[-1]) \rightarrow F(\varprojlim T_n) \rightarrow \varprojlim F(T_n) \rightarrow 0.$$

Therefore there exists $t(i) \in F(T(i)) = F(\varprojlim T_n)$ which maps in $(t_n)_{n \in \mathbb{N}}$ via the surjective morphism above. Putting $T = \prod_{i \in I} T(i)$ and $t = (t(i))_{i \in I}$ we want to show that (T, t) is a weakly initial object in \mathcal{K}/F . In order to prove this, consider an object $X \in \mathcal{K}$. By hypothesis, there is an inverse tower

$$X_0 \xleftarrow{u_0} X_1 \xleftarrow{u_1} T_2 \longleftarrow \dots$$

whose homotopy limit is X such that $X_0 \in \text{Prod}_0(\mathcal{S})$, and every X_{n+1} lies in a triangle $P_n \rightarrow X_{n+1} \xrightarrow{u_n} X_n \rightarrow P_n[1]$, for some $P_n \in \text{Prod}_0(\mathcal{S})$. We use again [1, Lemma 5.8(2)] for constructing the commutative diagram with

exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varprojlim^{(1)} \mathcal{K}(T, X_n[-1]) & \longrightarrow & \mathcal{K}(T, \varprojlim X_n) & \longrightarrow & \varprojlim \mathcal{K}(T, X_n) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \varprojlim^{(1)} F(X_n[-1]) & \longrightarrow & F(\varprojlim X_n) & \longrightarrow & \varprojlim F(X_n) \longrightarrow 0
\end{array}$$

whose columns are induced by the natural transformations which correspond to $t \in F(T)$ under Yoneda Lemma. If we would show that the two extreme vertical arrows are surjective, the same is true for the middle arrow too, and we are done. But for the first vertical map this follows by the commutative diagram:

$$\begin{array}{ccc}
\prod_{n \in \mathbb{N}} \mathcal{K}(T, X_n[-1]) & \longrightarrow & \varprojlim^{(1)} \mathcal{K}(T, X_n[-1]) \\
\downarrow & & \downarrow \\
\prod_{n \in \mathbb{N}} F(X_n[-1]) & \longrightarrow & \varprojlim^{(1)} F(X_n[-1])
\end{array}$$

whose arrows connected with the south-west corner are both surjective.

In order to prove the surjectivity of the third vertical map above we consider an element $x \in \varprojlim F(X_n)$, that is $x = (x_n)_{n \in \mathbb{N}} \in \prod F(X_n)$ such that $F(u_n)(x_{n+1}) = x_n$, for all $n \in \mathbb{N}$. Next we construct a commutative diagram

$$\begin{array}{ccccccc}
T_0 & \xleftarrow{w_0} & T_1 & \xleftarrow{w_1} & T_2 & \xleftarrow{\quad} & \dots \\
f_0 \downarrow & & f_1 \downarrow & & f_2 \downarrow & & \\
X_0 & \xleftarrow{u_0} & X_1 & \xleftarrow{u_1} & X_2 & \xleftarrow{\quad} & \dots
\end{array}$$

whose upper line is a tower in I , and satisfying $F(f_n)(t_n) = x_n$ for all $n \in \mathbb{N}$. This construction is performed inductively as follows: f_0 comes from the fact that (T_0, t_0) is weak initial in $\text{Prod}_n(\mathcal{S})/F$. Suppose the first n steps are done. We construct the following commutative diagram whose rows are triangles and the middle square is homotopy pull-down (see [14, Definition 1.4.1]):

$$\begin{array}{ccccccc}
P_n & \longrightarrow & Y_{n+1} & \longrightarrow & T_n & \longrightarrow & P_n[1] \\
\parallel & & \downarrow & & \downarrow f_n & & \parallel \\
P_n & \longrightarrow & X_{n+1} & \xrightarrow{u_n} & X_n & \longrightarrow & P_n[1]
\end{array}$$

The upper triangle shows that $Y \in \text{Prod}_{n+1}(\mathcal{S})$. Now Y_{n+1} is obtained via a triangle $Y_{n+1} \rightarrow T_n \times X_{n+1} \xrightarrow{(f_n, -u_n)} X_n \rightarrow Y_{n+1}[1]$. Applying the homological functor F we get an exact sequence:

$$F(Y_{n+1}) \rightarrow F(T_n) \times F(X_{n+1}) \xrightarrow{(F(f_n), -F(u_n))} F(X_n).$$

Since $F(f_n)(t_n) - F(u_n)(x_{n+1}) = x_n - x_n = 0$ we get an element $y_{n+1} \in Y_{n+1}$, which maps in (t_n, x_{n+1}) , via the first morphism of the exact sequence above. Since (T_{n+1}, t_{n+1}) is weak initial in $\text{Prod}_n(\mathcal{S})/F$ we find a map $(T_{n+1}, t_{n+1}) \rightarrow (Y_{n+1}, y_{n+1})$. The morphism f_{n+1} is the composition $T_{n+1} \rightarrow Y_{n+1} \rightarrow X_{n+1}$. The upper row above is, as noticed, an inverse tower in I , and

let denote it by i . Finally the element $t \in T$ maps to $(x_n)_{n \in \mathbb{N}} \in \varprojlim F(X_n)$, via the map $F(T) \rightarrow F(T(i)) \rightarrow \varprojlim F(T_n) \rightarrow \varprojlim F(X_n)$, proving that the map $\varprojlim \mathcal{K}(T, X_n) \rightarrow \varprojlim F(X_n)$ is surjective. \square

Combining Theorem 1.3 and Lemma 2.1 we obtain:

Theorem 2.2. *Let \mathcal{K} be a triangulated category and let $\mathcal{S} \subseteq \mathcal{K}$ be a set. Suppose that every $X \in \mathcal{K}$ is \mathcal{S} -cofiltered. Then every homological product preserving functor $F : \mathcal{K} \rightarrow \mathcal{A}b$ is representable, therefore \mathcal{K}^o satisfies Brown representability.*

3. BROWN REPRESENTABILITY FOR THE DUAL OF A HOMOTOPY CATEGORY

Throughout of this section \mathcal{A} , will denote an additive category, that is preadditive, with zero object and finitely biproducts; we suppose also that \mathcal{A} has splitting idempotents. We consider categories $\mathbf{C}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$ called the *category of complexes* respectively the *homotopy category of complexes* over \mathcal{A} , both of them having as objects complexes of objects in \mathcal{A} , that is a chain of objects and morphisms (called *differentials*) in \mathcal{A} of the form

$$X = \cdots \rightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \rightarrow \cdots,$$

such that $d_X^n d_X^{n-1} = 0$ for all $n \in \mathbb{Z}$. The morphisms in the category $\mathbf{C}(\mathcal{A})$ are families $(f^n)_{n \in \mathbb{Z}}$ of morphisms in \mathcal{A} commuting with differentials, and $\mathbf{K}(\mathcal{A})(X, Y) = \mathbf{C}(\mathcal{A})(X, Y) / \sim$ where \sim is an equivalence relation called *homotopy*, defined as follows: two maps of complexes $(f^n)_{n \in \mathbb{Z}}, (g^n)_{n \in \mathbb{Z}} : X \rightarrow Y$ are homotopically equivalent if there is $s^n : X^n \rightarrow Y^{n-1}$, for all $n \in \mathbb{Z}$ such that $f^n - g^n = d_Y^{n-1} s^n + s^{n+1} d_X^n$. Then it is well-known that $\mathbf{K}(\mathcal{A})$ is a triangulated category, possessing (co)products provided \mathcal{A} does the same. Note also that $\mathbf{C}(\mathcal{A})$ is an exact category (in the sense of [6, Section 4]) with respects short exact sequences which split in each degree (see [6, Example 4.3]), and $\mathbf{K}(\mathcal{A})$ may be constructed as the stable category of this exact category by [6, Example 6.1].

Fix the additive category \mathcal{A} as before. For an object $G \in \mathcal{A}$ we denote by $\text{Prod}(G)$ respectively $\text{Add}(G)$ the full subcategory consisting of direct factors (or equivalently, direct summands) of a product (respectively coproduct) of copies of G (assuming that the requested products or coproducts exist). We say that \mathcal{A} has a *product generator* if there is an object $G \in \mathcal{A}$ such that $\mathcal{A} = \text{Prod}(G)$. For the dual situation when $\mathcal{A} = \text{Add}(G)$ we use the more standard terminology \mathcal{A} is *pure-semisimple* (see [18, Definition 2.1 and Proposition 2.2]).

Lemma 3.1. *If \mathcal{A} is an additive category with splitting idempotents and products, which possesses, in addition, a product generator G . Denote $\mathcal{S} = \{G[n] \mid n \in \mathbb{Z}\}$ the closure of G under suspensions and desuspensions.*

- a) *If given two composable maps $X \rightarrow Y \rightarrow Z$ whose composition is 0 in \mathcal{A} , then $X \rightarrow Y$ factors through a subobject $Y' \leq Y$ such that the composed map $Y' \rightarrow Y \rightarrow Z$ vanishes, then $\mathbf{K}(\mathcal{A})$ is \mathcal{S} -cofiltered.*
- b) *If \mathcal{A} has images or kernels, then $\mathbf{K}(\mathcal{A})$ is \mathcal{S} -cofiltered.*

Proof. a) We will show inductively that a bounded complex with less than n non-zero entries is in $\text{Prod}_n(\mathcal{S})$. This is clear for $n = 0$. Now we suppose the property true for any complex with less than $n - 1$ non-zero entries. Let

$$\cdots \rightarrow 0 \rightarrow X^0 \rightarrow \cdots \rightarrow X^n \rightarrow 0 \rightarrow \cdots$$

be a bounded complex. The diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & X^n & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow = & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & X^0 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow = & & & & \downarrow = & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & X_0 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

is an exact sequence of complexes which splits in each degree. According to [6, Example 6.1] it leads to a triangle proving the induction step.

Finally consider an infinite complex

$$X = \cdots \rightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \rightarrow \cdots.$$

By hypothesis, the map d^{n-1} factors through a subobject $Y^n \leq X^n$, such that $Y^n \rightarrow X^n \xrightarrow{d^n} X^{n+1}$ vanishes, for all $n \in \mathbb{Z}$. For all $i \in \mathbb{N}$, consider the bounded complex

$$X(i) = \cdots \rightarrow 0 \rightarrow Y^{-i} \rightarrow X^{-i} \rightarrow X^{-i+1} \rightarrow \cdots \rightarrow X^{i-1} \rightarrow X^i \rightarrow 0 \rightarrow \cdots,$$

and the map of complexes $\epsilon(i) : X(i+1) \rightarrow X(i)$ as in the following diagram:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & Y^{-i} & \longrightarrow & X^{-i} & \longrightarrow & \cdots & \longrightarrow & X^i & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \parallel & & & & \parallel & & \uparrow & & \\ \cdots & \longrightarrow & Y^{-i-1} & \longrightarrow & X^{-i-1} & \longrightarrow & X^{-i} & \longrightarrow & \cdots & \longrightarrow & X^i & \longrightarrow & X^{i+1} & \longrightarrow & \cdots \end{array}$$

Applying [5, Lemma 2.6] we infer that X is isomorphic in $\mathbf{K}(\mathcal{A})$ to the homotopy limit of a chain of bounded complexes

$$\cdots \rightarrow X(2) \xrightarrow{\epsilon(1)} X(1) \xrightarrow{\epsilon(0)} X(0),$$

thus X is \mathcal{S} -cofiltered.

b) We apply a) with $Y^n = \text{im } d^{n-1}$ or $Y^n = \ker d^n$, for all $n \in \mathbb{Z}$. \square

Theorem 3.2. *If \mathcal{A} is an additive category with splitting idempotents, possessing images or kernels, then $\mathbf{K}(\mathcal{A})^o$ satisfies Brown representability if and only if \mathcal{A} has a product generator. In particular, if R is a ring, then $\mathbf{K}(\text{Mod}(R))^o$ satisfies Brown representability if and only if $\text{Mod}(R)$ has a product generator.*

Proof. The direct implication is [13, Theorem 2], whereas the converse follows by Lemma 3.1 b) and Theorem 2.2. Now the category $\text{Mod}(R)$ is additive and has both images and kernels. \square

Remark 3.3. If the ring R is pure-semisimple, then $\text{Mod}(R) = \text{Add}(G)$ for some $G \in \text{Mod}(R)$ (in fact G is the direct sum of a family of representants of all isomorphism classes of finitely presentable modules). In this case,

$\text{Add}(G)$ is closed under products, so G is product-complete hence $\text{Add}(G) = \text{Prod}(G)$ (see [7, Theorem 6.7]). Consequently $\mathbf{K}(\text{Mod}(R))^\circ$ satisfies Brown representability, by the Theorem above. This was already known since $\text{Mod}(R)$ is a pure-semisimple finitely presentable category which is closed under products, so it is compactly generated by [18, Theorem 5.2]. It would be therefore interesting to characterize ring R for which the module category $\text{Mod}(R)$ has a product generator. If we could indicate a non pure-semisimple ring belonging to this class, then we would produce an example of a triangulated category with products and coproducts, namely $\mathcal{K} = \mathbf{K}(\text{Mod}R)$ such that \mathcal{K}° satisfies Brown representability, but \mathcal{K} does it not. Such an example is, at the best of our knowledge, unknown.

Remark 3.4. There is an isomorphism of categories $\mathbf{K}(\mathcal{A})^\circ \xrightarrow{\sim} \mathbf{K}(\mathcal{A}^\circ)$, which is easy to establish (for example, this is write down in [11, Theorem 2.1.1]). Applying this isomorphism of categories, we may dualize all results in this section. Thus we may conclude that if \mathcal{A} is an additive category with splitting idempotents possessing images or cokernels, then $\mathbf{K}(\mathcal{A})$ satisfies Brown representability theorem if and only if \mathcal{A} is pure-semisimple. Note that this statement is already known for $\mathcal{A} = \text{Mod}(R)$, or more generally for \mathcal{A} a finitely presented category with coproducts, as we may see by a combination between [13, Theorem 1] and [18, Proposition 2.6]. However the results in [13] and [18] may not be dualized in order to obtain the above section, since the argument used there for showing that $\mathbf{K}(\mathcal{A})$ satisfies Brown representability, where \mathcal{A} is a pure-semisimple, finitely presentable additive category with coproducts goes as follows: If \mathcal{A} enjoys all these properties, then $\mathbf{K}(\mathcal{A})$ is well generated by [18, Theorem 5.2], therefore it satisfies Brown representability by [14, Theorem 8.3.3 and proposition 8.4.2]. But no one of the notions “module category”, “finitely presentable category” and “well generated triangulated category” is self-dual.

Remark 3.5. Let R be a ring with $\text{gldim } R \leq 2$. Then the category $\text{Inj-}R$ of all injective modules is additive, closed under products, idempotents and images and every injective cogenerator of $\text{Mod}(R)$ is a product generator for $\text{Inj-}R$. Thus Theorem 3.2 gives another proof for the fact that $\mathbf{K}(\text{Inj-}R)^\circ$ satisfies Brown representability, fact which is also known since $\mathbf{K}(\text{Inj-}R)$ is equivalent to the derived category which is compactly generated.

Example 3.6. In Introduction we said that this paper complete the picture in [12]. Note that [12, Theorem 3] gives an example of a triangulated coproduct preserving functor which has no right adjoint. Using the equivalence of categories $\mathbf{K}(\mathcal{A})^\circ \xrightarrow{\sim} \mathbf{K}(\mathcal{A}^\circ)$ from Remark 3.4, we obtain a triangulated product preserving functor which has no left adjoint.

Here we will provide another example of this kind, which holds only in an extension of ZFC. More precisely, assume there are no measurable cardinals. For every cardinal λ let denote by \mathbb{Z}^λ the product of λ -copies of \mathbb{Z} and by $\mathbb{Z}^{<\lambda}$ its subgroup consisting of sequences with support (i.e. the set of non-zero entries) of cardinality smaller than λ . Let $\mathcal{A} \subseteq \mathcal{Ab}$ be the closure under products and direct factors of the class of all abelian groups of the form $\mathbb{Z}^\lambda / \mathbb{Z}^{<\lambda}$, where λ runs over all cardinals. The inclusion functor $\mathbf{K}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{Ab})$ is triangulated and preserves products. If we suppose that

it has a left adjoint then $\mathbf{K}(\mathcal{A})$ must be preenveloping in $\mathbf{K}(\mathcal{A}b)$ by Corollary 1.6. For $A \in \mathcal{A}b$, the complex having X in degree 0 and 0 elsewhere must have an $\mathbf{K}(\mathcal{A})$ -preenvelope, which is a complex X with entries in \mathcal{A} . It is not hard to see that $X \rightarrow X^0$ is an \mathcal{A} -preenvelope on A . But this contradicts [2, Proposition 2.5], where it is shown that, under the hypothesis of nonexistence of measurable cardinals, the class \mathcal{A} is not preenveloping in $\mathcal{A}b$.

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BABEȘ-BOLYAI UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE,
1, MIHAIL KOGĂLNICEANU, 400084 CLUJ-NAPOCA, ROMANIA
E-mail address: cmodoi@math.ubbcluj.ro